AIAA JOURNAL
Vol. 39, No. 12, December 2001

Anisotropic Thin-Walled Beams with Closed Cross-Sectional Contours

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A complete theory of free torsion and bending of anisotropic thin-walled beams with closed cross-sectional contours is developed using the stress formulation. In contrast to a displacement formulation that is traditionally used for this purpose, the stress formulation does not require the introduction of assumptions concerning the form of the warping function. The advantage of the stress formulation is the exact satisfaction of the equilibrium equations, which provides more accurate results for beams whose stiffness coefficients depend on the contour coordinate, for example, beams stiffened with stringers. Approximate compatibility conditions obtained with the aid of Castigliano's theorem describe all possible types of coupling effects and give explicit expressions for displacements of a beam with an arbitrary cross-sectional contour and material anisotropy. Theoretical results are compared to experimental data, and fair agreement is found.

Introduction

COMPOSITE thin-walled beams have been under extensive study for about a decade because of their potential to enhance the performance of helicopter rotor blades, propellers, and wing structures of advanced aircraft. Anisotropic beams are particularly attractive because of the material coupling effects that can be used for structural tailoring,[1] providing new links between the structural stiffness and aerodynamic characteristics. Surveys of anisotropic beam theory have been published by Hodges,[2] Rehfield et al.,[3] and Smith and Chopra.[4]

In general, the problems of thin-walled beam theory can be divided into two classes. The first class of problems is the so-called free warping of the cross section, in which warping refers to the displacement of the cross section out of its flexural plane. The second class problems is restrained warping of the cross section as occurs when the ends of the beam are prevented from warping displacement by boundary conditions. Restrained warping effects are significant in short beams or in narrow boundary layers at the ends of long beams where the kinematic restraint is imposed. The subject of this paper is free warping of anisotropic beams, and restrained warping is not discussed. Important aspects of the anisotropic material couplings are manifest in the study of free warping of beams subjected to extension, bending, and torsion.

The problem of beam deformation under free warping can be analyzed by a displacement formulation or by stress formulations. Displacement formulations are based on some approximation for the displacement distribution along the contours of the section, for example, see Smith and Chopra.[4] This means the warping component of the axial displacement is preassigned independent of material properties. The principal shortcoming of the displacement formulation follows from that the shear flow (shear stress tangent to the contour multiplied by the wall thickness) is found from the constitutive equation and is proportional to the shear stiffness of the wall. This restricts the application of this formulation to beams whose shear stiffness does not depend on the contour coordinate, or to beams with slowly varying shear stiffness in the contour coordinate. If this is not the case, displacement formulations become inconsistent and, in limiting cases, incorrect. For example, if a beam panel has infinitely high shear stiffness (which is the model for a stringer flange), then the shear flow in this panel becomes infinitely high and so does the shear stiffness of the whole beam.

In contrast to displacement formulations of the problem, the stress formulation, which is well known for isotropic beams and is described for composite beams by Vasiliev,[5] does not require a priori assumptions for the warping function. The shear flow is determined from an equilibrium equation by integration that does not impose any restrictions on the manner in which the wall stiffness varies along the cross-sectional contour. Thus, the stress formulation results in a more general and more accurate expression for the shear flow. For anisotropic beams, this expression is presented by Libove.[6] To find the displacements, we need to impose compatibility conditions that can be satisfied in the stress formulation only approximately with the aid of a variational theorem. However, though more rigorous analytical models[7] and numerical finite element models[8,9] have been developed recently for composite thin-walled beams, the classical stress formulation applied hereafter to anisotropic beams, with an arbitrary form of the cross section and material distribution along its contour, leads to a closed-form solution suitable for design problems.

Governing Equations

Consider a thin-walled beam fixed at the cross section $z = 0$ with respect to cross-sectional displacements and rotations as a solid (cross-sectional warping is not restrained, however) and loaded at the end $z = L$ with and axial force P_L, transverse forces Q_x^L and Q_y^L, torque T_L, and bending moments M_x^L and M_y^L as is depicted in Fig. 1. Then equilibrium equations of the beam element provide the following expressions for the forces and moments acting in an arbitrary cross section $z = \text{const}$:

$$P = P_L, \qquad Q_x = Q_x^L, \qquad Q_y = Q_y^L, \qquad T = T_L$$
$$M_x = M_x^L - Q_y^L(L - z), \qquad M_y = M_y^L - Q_x^L(L - z) \quad (1)$$

The differential equations for equilibrium of an infinitesimal element of the beam (see Fig. 1) are

Received 20 August 1998; revision received 4 June 1999; accepted for publication 1 June 2001.

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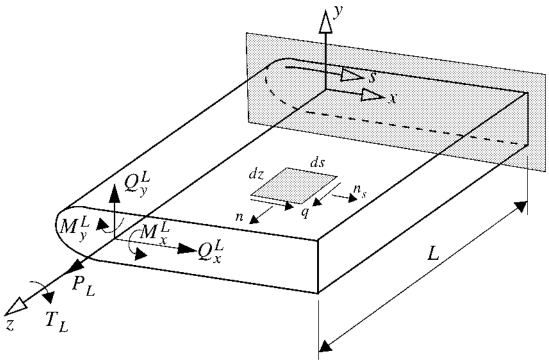

Fig. 1 Forces and moments acting on beam.

$$n' + \dot{q} = 0, \qquad q' + \dot{n}_s = 0, \qquad \kappa_s n_s = 0 \tag{2}$$

where partial derivatives are denoted as $()' = \partial()/\partial z$ and $\dot{()} = \partial()/\partial s$, κ_s is the curvature of the cross-sectional contour, n is the normal stress resultant in the axial direction, n_s is the normal stress resultant in the contour direction, and q is the shear stress resultant, or shear flow.

Constitutive equations for an anisotropic material can be written as

$$\begin{bmatrix} n \\ n_s \\ q \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \varepsilon_s \\ \gamma \end{bmatrix} \tag{3}$$

Here, the quantities $B_{ij}\,(=B_{ji})$ are stiffness coefficients specified for laminated composite wall construction as given by Vasiliev[5]; ε and ε_s are axial and contour normal strains, respectively; and γ is the shear strain.

Assumptions

To formulate the first assumption, consider the last equation of Eqs. (2), which yields $n_s = 0$ for $\kappa_s \neq 0$. Extending this result for a general form of the cross section, we assume that $n_s = 0$ for the theory under study. Also, this assumption is made in classical beam theory where lateral stresses are neglected with respect to the axial normal stress in the material law. Assuming $n_s = 0$ permits the solution of the contour normal strain ε_s from Eqs. (3), and then permits its elimination in the remaining two constitutive equations. The following two forms will be used in subsequent developments:

$$n = B\varepsilon + bq, \qquad q = B_s\gamma + B_z\varepsilon \tag{4}$$

$$\varepsilon = (1/B)(n - bq), \qquad \gamma = (1/B)(aq - bn) \tag{5}$$

where

$$B = B_{11} - B_{12}^2\big/B_{22} - bB_z \tag{6a}$$

$$B_s = B_{33} - B_{23}^2\big/B_{22}, \qquad B_z = B_{13} - B_{12}B_{23}/B_{22} \tag{6b}$$

$$a = (1/B_s)\big(B_{11} - B_{12}^2\big/B_{22}\big), \qquad b = B_z/B_s \tag{6c}$$

The second assumption is traditional for the beam theory and states that the axial strain is linear function of x and y, that is,

$$\varepsilon = W' + x\theta_y' + y\theta_x' \tag{7}$$

In this equation, $W(z)$ is the axial displacement of the cross section, and $\theta_x(z)$ and $\theta_y(z)$ are the angles of rotation of the cross section about the x and y axes, respectively.

Finally, we assume that the displacements of any point on the contour in the cross-sectional plane that are associated with deformation of the contour are small in comparison with the corresponding displacements of the cross section of the beams as a rigid disk. As a result, we can write the displacements of any point of the contour in the x and y directions as

$$u_x = U + y\theta, \qquad u_y = V - x\theta \tag{8}$$

where $U(z)$ and $V(z)$ are displacements of the origin O in the x and y directions, or the deflections of the beam axis in xz and yz planes, and $\theta(z)$ is the twist angle, or angle of rotation of the cross section about the z axis. Because the slopes of the beam axis in xz and yz planes, which are represented by the derivatives U' and V', and the rotations of the cross section θ_x and θ_y, are independent functions, we can conclude that the cross sections do not remain orthogonal to the axis of the beam in the deformed state. Thus, there should exist transverse shear deformations of the beam that are

$$\Psi_x = U' + \theta_y, \qquad \Psi_y = V' + \theta_x \tag{9}$$

in the xz and yz planes, respectively. Indeed, putting $\Psi_x = 0$ and $\Psi_y = 0$, we get $\theta_y = -U'$ and $\theta_x = -V'$, which correspond to the classical Bernoulli–Euler beam theory.

Normal and Shear Stress Resultants

First consider the axial stress resultant n specified by the first of Eqs. (4). Substituting ε from Eq. (7), we get

$$n = B(W' + x\theta_y' + y\theta_x') + bq \tag{10}$$

At any cross section of the beam, the distribution of the resultant n can be reduced to an axial force P and bending moments M_x and M_y, as shown in Fig. 1. The corresponding conditions of static equivalence are

$$P = \oint n\,\mathrm{d}s, \qquad M_x = \oint ny\,\mathrm{d}s, \qquad M_y = \oint nx\,\mathrm{d}s \tag{11}$$

Substitution of Eq. (10) yields three equations for W', θ_x', and θ_y' whose solution can be written as

$$W' = \bar{P}/S - x_0\theta_y' - y_0\theta_x'$$

$$\theta_x' = \big(k\big/D_x^o\big)[\bar{M}_x - y_0\bar{P} - k_y(\bar{M}_y - x_0\bar{P})]$$

$$\theta_y' = \big(k\big/D_y^o\big)[\bar{M}_y - x_0\bar{P} - k_x(\bar{M}_x - y_0\bar{P})] \tag{12}$$

where

$$\bar{P} = P - \oint bq\,\mathrm{d}s, \qquad \bar{M}_x = M_x - \oint bqy\,\mathrm{d}s$$

$$\bar{M}_y = M_y - \oint bqx\,\mathrm{d}s \tag{13}$$

Equations (12) include the following characteristics of the cross section of the beam:

$$x_0 = S_y/S, \qquad y_0 = S_x/S \tag{14a}$$

$$k = 1/(1 - k_xk_y), \qquad k_x = D_{xy}^o\big/D_x^o, \qquad k_y = D_{xy}^o\big/D_y^o \tag{14b}$$

$$S = \oint B\,\mathrm{d}s, \qquad S_x = \oint By\,\mathrm{d}s, \qquad S_y = \oint Bx\,\mathrm{d}s \tag{14c}$$

$$D_x^o = D_x - y_0^2S, \qquad D_y^o = D_y - x_0^2S, \qquad D_{xy}^o = D_{xy} - x_0y_0S \tag{14d}$$

$$D_x = \oint By^2\,\mathrm{d}s, \qquad D_y = \oint Bx^2\,\mathrm{d}s, \qquad D_{xy} = \oint Bxy\,\mathrm{d}s \tag{14e}$$

where x_0 and y_0 are the coordinates of the modulus-weighted centroid of the cross section, S is the axial stiffness of the beam, and D_x^o and D_y^o are the bending stiffnesses of the beam referred to the centroidal coordinates. (If $D_{xy}^o = 0$, they become principal coordinates of the cross section.)

To proceed, we need to determine the shear stress resultant, or shear flow q, which enters Eqs. (10). Consider Eqs. (2) first. Because $n_s = 0$, the second equation yields $q' = 0$, which means that the shear flow does not depend on coordinate z. Now we can use the first equilibrium equation in Eqs. (2) to find q. Substituting n from Eq. (10), and W', θ_x', and θ_y', from Eqs. (12), we arrive at

$$\dot{q} = -Bk\left[\left(Q_y/D_x^o\right)\bar{y} + \left(Q_x/D_y^o\right)\bar{x}\right] \tag{15}$$

where

$$\bar{x} = x - x_0 - k_x(y - y_o), \qquad \bar{y} = y - y_0 - k_y(x - x_0)$$

The right-hand side of Eq. (15) does not contain q, because $q' = 0$, and it is transformed with the aid of equations $P' = 0$, $M_x' = Q_y$, and $M_y' = Q_x$ following from Eqs. (1). Integrating Eq. (15), we get

$$q(s) = -k\left[\left(Q_y/D_x^o\right)S_x(s) + \left(Q_x/D_y^o\right)S_y(s)\right] + q_0 \tag{16}$$

where

$$S_x(s) = \int_0^s B\bar{y}\,\mathrm{d}s, \qquad S_y(s) = \int_0^s B\bar{x}\,\mathrm{d}s \tag{17}$$

As can be shown,[5] the shear flow distribution given by Eq. (16) is statically equivalent to the shear forces Q_x and Q_y. Thus, the only static condition that is not satisfied to this point is the equivalence of the shear flow to the torque T acting in the cross section. Let $r(s)$ denote the perpendicular distance from the origin O to the tangent to the contour at the point with contour coordinate s. Then, the corresponding equation of statical torsional equivalence is

$$T = \oint rq\,\mathrm{d}s$$

which allows us to find the integration constant q_0 in Eq. (16) and to write this equation in the following final form:

$$q = Q_x F_x(s) + Q_y F_y(s) + T/2A \tag{18}$$

where

$$F_x(s) = -\frac{k}{D_y^o}\left[S_y(s) - \frac{1}{2A}\oint S_y(s) r\,\mathrm{d}s\right]$$

$$F_y(s) = -\frac{k}{D_x^o}\left[S_x(s) - \frac{1}{2A}\oint S_x(s) r\,\mathrm{d}s\right], \qquad A = \frac{1}{2}\oint r\,\mathrm{d}s$$

where $S_x(s)$ and $S_y(s)$ are specified by Eqs. (17) and A is the area enclosed by the cross-sectional contour. It can be shown that Eq. (18) determines the shear flow distribution that is independent of the contour origin, where at the contour origin $s = 0$. Also note that the shear flow depends only on the reduced stiffness B of the wall, which is specified by Eq. (6a).

Having found the shear flow q, we can substitute it into Eqs. (10) and (13) and determine the normal stress resultant. The final result is as follows:

$$n = B(P/S) + Bk\left[(M_x - y_0 P)\left(\bar{y}/D_x^0\right) + (M_y - x_0 P)\left(\bar{x}/D_y^o\right)\right]$$
$$+ Q_x\Phi_x(s) + Q_y\Phi_y(s) + (T/2A)\Phi(s) \tag{19}$$

where

$$\Phi_x(s) = bF_x(s) - \frac{B}{S}\oint bF_x(s)\,\mathrm{d}s - Bk\left[\frac{\bar{y}}{D_x^o}\oint bF_x(s)(y - y_0)\,\mathrm{d}s\right.$$
$$\left. + \frac{\bar{x}}{D_y^o}\oint bF_x(s)(x - x_0)\,\mathrm{d}s\right] \tag{20a}$$

$$\Phi_y(s) = bF_y(s) - \frac{B}{S}\oint bF_y(s)\,\mathrm{d}s - Bk\left[\frac{\bar{y}}{D_x^o}\oint bF_y(s)(y - y_0)\,\mathrm{d}s\right.$$
$$\left. + \frac{\bar{x}}{D_y^o}\oint bF_y(s)(x - x_0)\,\mathrm{d}s\right] \tag{20b}$$

$$\Phi(s) = b - \frac{B}{S}\oint b\,\mathrm{d}s - Bk\left[\frac{\bar{y}}{D_x^o}\oint b(y - y_0)\,\mathrm{d}s\right.$$
$$\left. + \frac{\bar{x}}{D_y^o}\oint b(x - x_0)\,\mathrm{d}s\right] \tag{20c}$$

Thus, shear and normal stress resultants in anisotropic beams are specified by Eqs. (18) and (19).

Displacements and Rotations of the Cross Section

Having found the stress resultants and having satisfied all of the equilibrium equations, we can use Castigliano's theorem to determine displacements and rotations of an arbitrary beam cross section. To do this, consider a beam element like that shown in Fig. 1, but of length $L = \mathrm{d}z$, and apply to its free end forces P, Q_x, and Q_y and moments M_x, M_y, and T. The strain energy of this element can be presented as

$$\mathrm{d}U = \frac{\mathrm{d}z}{2}\oint(n\varepsilon + q\gamma)\,\mathrm{d}s = \frac{\mathrm{d}z}{2}\oint\frac{1}{B}(n^2 - 2bnq + aq^2)\,\mathrm{d}s \tag{21}$$

The second part of this expression is derived using Eqs. (5). According to Castigliano's theorem, we can write the following relationships:

$$\mathrm{d}\theta_x = \frac{\partial}{\partial M_x}(\mathrm{d}U), \qquad \mathrm{d}\theta_y = \frac{\partial}{\partial M_y}(\mathrm{d}U), \qquad \mathrm{d}W = \frac{\partial}{\partial P}(\mathrm{d}U)$$

$$\Psi_x\,\mathrm{d}z = \frac{\partial}{\partial Q_x}(\mathrm{d}U), \qquad \Psi_y\,\mathrm{d}z = \frac{\partial}{\partial Q_y}(\mathrm{d}U), \qquad \mathrm{d}\theta = \frac{\partial}{\partial T}(\mathrm{d}U)$$

Substituting n and q from Eqs. (18) and (19) into Eq. (21), and then performing differentiation, we obtain the following set of equations:

$$\begin{bmatrix}\theta_x'\\ \theta_y'\\ W'\\ \Psi_x\\ \Psi_y\\ \theta'\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}M_x\\ M_y\\ P\\ Q_x\\ Q_y\\ T\end{bmatrix} \tag{22}$$

where

$$C_{11} = \frac{k}{D_x^o} \tag{23a}$$

$$C_{12} = C_{21} = -\frac{kk_y}{D_x^o} \tag{23b}$$

$$C_{13} = C_{31} = -\frac{k}{D_x^o}y_0 + \frac{kk_x}{D_y^o}x_0 \tag{23c}$$

$$C_{14} = C_{41} = -\frac{k}{D_x^o}\oint bF_x(s)\bar{y}\,\mathrm{d}s \tag{23d}$$

$$C_{15} = C_{51} = -\frac{k}{D_x^o}\oint bF_y(s)\bar{y}\,\mathrm{d}s \tag{23e}$$

$$C_{16} = C_{61} = -\frac{k}{2AD_x^o}\oint b\bar{y}\,\mathrm{d}s \tag{23f}$$

$$C_{22} = \frac{k}{D_y^o} \tag{23g}$$

$$C_{23} = C_{32} = -\frac{k}{D_y^o}x_0 + \frac{kk_y}{D_x^o}y_0 \tag{23h}$$

$$C_{24} = C_{42} = -\frac{k}{D_y^o}\oint bF_x(s)\bar{x}\,\mathrm{d}s \tag{23i}$$

$$C_{25} = C_{52} = -\frac{k}{D_y^o}\oint bF_y(s)\bar{x}\,\mathrm{d}s \tag{23j}$$

$$C_{26} = C_{62} = -\frac{k}{2AD_y^o}\oint b\bar{x}\,\mathrm{d}s \tag{23k}$$

$$C_{33} = \frac{1}{S} + \frac{k}{D_x^o}\left(y_0^2 - k_y x_0 y_0\right) + \frac{k}{D_y^o}\left(x_0^2 - k_x x_0 y_0\right) \tag{23l}$$

$$C_{34} = C_{43} = -\frac{1}{S}\oint bF_x(s)\,\mathrm{d}s + \frac{ky_o}{D_x^o}\oint bF_x(s)\bar{y}\,\mathrm{d}s + \frac{kx_0}{D_y^o}\oint bF_x(s)\bar{x}\,\mathrm{d}s \tag{23m}$$

$$C_{35} = C_{53} = -\frac{1}{S}\oint bF_y(s)\,\mathrm{d}s + \frac{kx_o}{D_y^o}\oint bF_y(s)\bar{x}\,\mathrm{d}s + \frac{ky_0}{D_x^o}\oint bF_y(s)\bar{y}\,\mathrm{d}s \tag{23n}$$

$$C_{36} = C_{63} = -\frac{1}{2AS}\oint b\,\mathrm{d}s + \frac{ky_0}{2AD_x^o}\oint b\bar{y}\,\mathrm{d}s + \frac{kx_0}{2AD_y^o}\oint b\bar{x}\,\mathrm{d}s \tag{23o}$$

$$C_{44} = \oint\frac{1}{B}\left[\Phi_x^2(s) - 2bF_x(s)\Phi_x(s) + aF_x^2(s)\right]\mathrm{d}s \tag{23p}$$

$$C_{45} = C_{54} = \oint\frac{1}{B}[\Phi_x(s)\Phi_y(s) - bF_x(s)\Phi_y(s) - b\Phi_x(s)F_y(s) + aF_x(s)F_y(s)]\,\mathrm{d}s \tag{23q}$$

$$C_{46} = C_{64} = \frac{1}{2A}\oint\frac{1}{B}[\Phi(s)\Phi_x(s) - b\Phi(s)F_x(s) - b\Phi_x(s) + aF_x(s)]\,\mathrm{d}s \tag{23r}$$

$$C_{55} = \oint\frac{1}{B}\left[\Phi_y^2(s) - 2bF_y(s)\Phi_y(s) + aF_y^2(s)\right]\mathrm{d}s \tag{23s}$$

$$C_{56} = C_{65} = \frac{1}{2A}\oint\frac{1}{B}[\Phi(s)\Phi_y(s) - b\Phi(s)F_y(s) - b\Phi_y(s) + aF_y(s)]\,\mathrm{d}s \tag{23t}$$

$$C_{66} = \frac{1}{4A^2}\oint\frac{1}{B}[\Phi^2(s) - 2b\Phi + a]\,\mathrm{d}s \tag{23u}$$

Thus, the equations obtained from Castigliano's theorem, Eqs. (22), include 21 independent coefficients C_{ij}, 15 of which (the off-diagonal coefficients) correspond to different types of coupling effects. Some of these coupling effects are of a geometric nature and can be eliminated by the proper choice of a coordinate frame, for example, referring the cross section to principal centroidal coordinates for which $x_0 = y_0 = 0$ and $k_x = k_y = 0$, we get $C_{12} = 0$, $C_{13} = 0$, and $C_{23} = 0$, so that bending–extension coupling is eliminated. The remaining 12 coefficients C_{ij} reflect bending–shearing coupling (C_{14}, C_{15}, C_{24}, and C_{25}), bending–torsion coupling (C_{16} and C_{26}), shearing–extension coupling (C_{34} and C_{35}), torsion–extensional coupling (C_{36}), torsional–shearing coupling (C_{46} and C_{56}), and coupling between transverse shear deformations in the xz and yz planes (C_{45}).

The solution of Eqs. (22) allows us to find θ_x', θ_y', and W' and Ψ_x', Ψ_y', and θ'. The first three equations (for θ_x', θ_y', and W') coincide with Eqs. (12) and can be derived from equilibrium conditions, whereas the last three equations (for Ψ_x', Ψ_y', and θ') are compatibility equations that do not follow from equilibrium conditions. Having the solution of Eqs. (22) and using Eqs. (9), we can find the rotation angles for any arbitrary cross section of the beam in Fig. 1, that is,

$$\theta_x = \int_0^z \theta_x'\,\mathrm{d}z, \qquad \theta_y = \int_0^z \theta_y'\,\mathrm{d}z, \qquad \theta = \int_0^z \theta'\,\mathrm{d}z \tag{24}$$

and displacements of this cross section as

$$W = \int_0^z W'\,\mathrm{d}z, \qquad U = \int_0^z (\Psi_x - \theta_y)\,\mathrm{d}z, \qquad V = \int_0^z (\Psi_y - \theta_x)\,\mathrm{d}z \tag{25}$$

Note that, although the foregoing derivation is exactly valid for a beam loaded at the ends, the results can be extended for beams loaded with transverse distributed force intensities. Stress resultants and displacements are determined with the aid of Eqs. (18), (19), (24), and (25), in which the bending moments, transverse forces, and the torque should be found from the following equations:

$$Q_x' + p_x = 0, \qquad Q_y' + p_y = 0, \qquad M_x' - Q_y + m_x = 0$$
$$M_y' - Q_x + m_y = 0, \qquad T' + m_z = 0 \tag{26}$$

where p_x and p_y denote distributed line load intensities acting along the z axis in the x and y directions, respectively, which are due to surface or body forces, and m_x, m_y, and m_z denote distributed line moment intensities acting along the z axis directed around the x, y, and z axes, respectively.

Examples

To demonstrate the most important coupling effects, consider beams with circular and rectangular cross sections. First, we examine an anisotropic circular beam whose stiffness parameters B, b, and a, entering Eqs. (5), do not depend on the contour coordinate s. For this case, Eqs. (22) acquire the form

$$\theta_x' = C_{11}M_x + C_{14}Q_x, \qquad \Psi_x = C_{41}M_x + C_{44}Q_x$$
$$\theta_y' = C_{22}M_y + C_{25}Q_y, \qquad \Psi_y = C_{52}M_x + C_{55}Q_y$$
$$W' = C_{33}P + C_{36}T, \qquad \theta' = C_{63}P + C_{66}T \tag{27}$$

where

$$C_{11} = C_{22} = 1/\pi R^3 B, \qquad C_{14} = -C_{25} = -(b/\pi R^2 B)$$
$$C_{33} = 1/2\pi RB, \qquad C_{36} = -(b/2\pi R^2 B)$$
$$C_{44} = C_{55} = a/\pi RB, \qquad C_{66} = a/2\pi R^3 B$$

in which R is the radius of the cross-sectional contour.

For a graphite–epoxy composite beam experimentally studied by Nixon,[10] dependence of the twist rate θ' on the applied torque T is shown in Fig. 2, where curve 1 corresponds to a beam without axial load and curve 2 corresponds to a beam with torsional–extensional coupling subjected to an axial tensile force $P_L = 4.45$ kN. The beam in the experiment has a radius $R = 20.83$ mm, wall thickness 1.016 mm, and is fabricated from unidirectional tape with a wall layup of $[(20/-70)_2]_s$. (Note that the twist rate and torque are defined counterclockwise positive in the experiment, which is opposite to the definition in this paper.)

As a second example, consider the box beam studied by Smith and Chopra[4] and Chandra et al.[11] The test article is a cantilevered, orthotropic ($b = 0$) cross-ply graphite–epoxy beam 762 mm long,

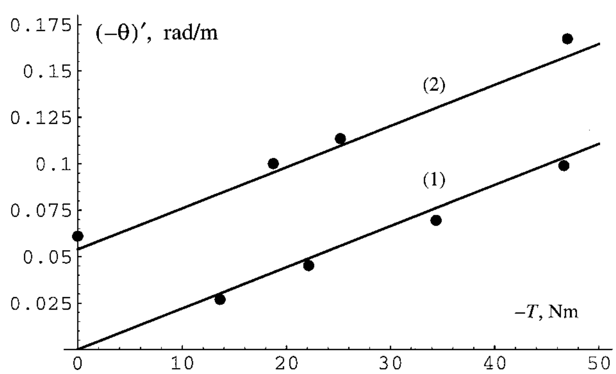

Fig. 2 Dependence of twist rate on torque for circular anisotropic beam without axial force (line 1) and under axial force P_L = 4.45 kN (line 2): ——, analysis; and •, experiment.[10]

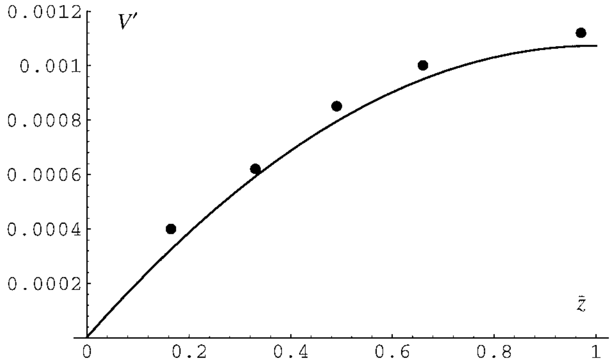

Fig. 3 Dependence of beam slope on axial coordinate for orthotropic box beam under transverse bending: ——, analysis; and •, experiment.[4]

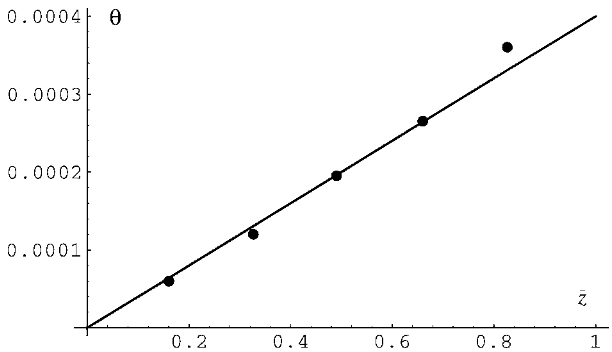

Fig. 4 Dependence of twist angle on axial coordinate for orthotropic box beam under torsion: ——, analysis; and •, experiment.[4]

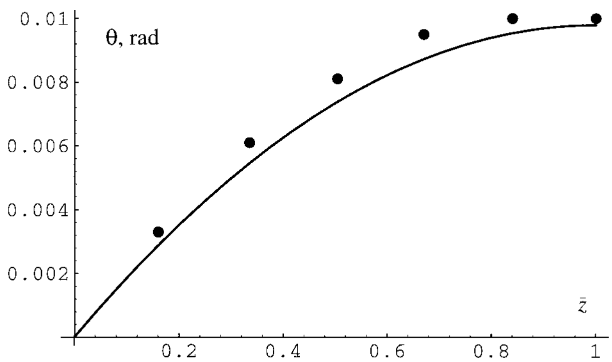

Fig. 5 Dependence of twist angle on axial coordinate for anisotropic box beam under transverse bending: ——, analysis; and •, experiment.[4]

with a cross-sectional width-by-height of 52.3 × 26 mm and a wall thickness of 0.76 mm. It is subjected to tip force $Q_y^L = 4.45$ N (1 lb) and torque $T_L = 0.113$ N · mm (1 lb · in.) (see Fig. 1). The response, as measured by the slope and twist angle with respect to the dimensionless axial coordinate $\bar{z} = z/L$, is shown in Figs. 3 and 4. To demonstrate coupling effects, consider the experimental box beam made of graphite–epoxy composite with a length of 762 mm, a cross-sectional width-by-height of 24.2 × 13.6 mm, and a wall thickness of 0.76 mm. The laminate in the upper horizontal panel is $[+15]_6$, for the lower horizontal panel it is $[-15]_6$, and the vertical panels are $[15/-15]_3$ balanced angle ply laminates. The specific feature of this beam is that anisotropy parameter b in Eq. (5) is positive for the upper horizontal panel, negative for the lower horizontal panel, and zero for the vertical panels. For this case, Eqs. (22) become

$$\theta_x' = C_{11}M_x + C_{16}T, \qquad \Psi_x = C_{43}P + C_{44}Q_x + C_{46}T$$

$$\theta_y' = C_{22}M_y, \qquad \Psi_y = C_{55}Q_y$$

$$W' = C_{33}P + C_{34}Q_x, \qquad \theta' = C_{61}M_x + C_{64}Q_x + C_{66}T$$

The twist angle induced by the force $Q_y^L = 4.45$ N (1 lb) is shown in Fig. 5, and the slope of the beam axis occurring under torsion with torque $T_L = 0.113$ N · mm (1 lb · in.) is presented in Fig. 6.

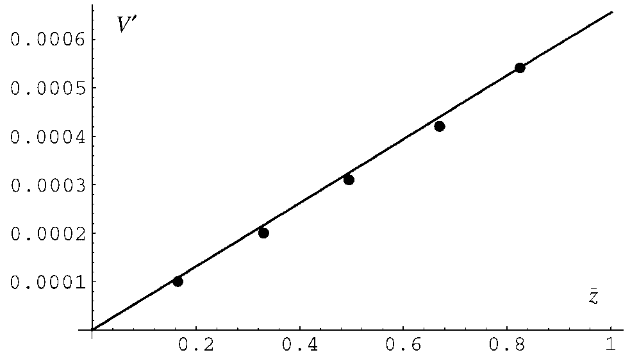

Fig. 6 Dependence of slope on axial coordinate for anisotropic box beam under torsion: ——, analysis; and •, experiment.[11]

Conclusions

As follows from the foregoing discussions and results, the stress formulation of thin-walled beam theory provides solutions allowing us to describe the behavior of anisotropic composite beams with reasonable accuracy. However, fair agreement with these and similar experiments is demonstrated by many published theories of anisotropic thin-walled beams, and the purpose of this paper was not to construct a more accurate theory. The purpose was to derive closed-form expressions specifying stresses and displacements of an anisotropic beam with an arbitrary cross section and material distribution along its contour. Real thin-walled beam structures can be composed of both isotropic and anisotropic panels and have longitudinal stiffeners (stringers). As can be seen from Eqs. (14), (17), (20), and (23), which specify stiffness and compliance coefficients, we can take into account any discontinuities or singularities of the beam wall stiffness.

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